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Unital Groups and General Comparability Property

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Pseudo-effect algebras are partial algebras $(E; +, 0, 1)$ with a partially defined addition $+$ which is not necessarily commutative and therefore with two complements, left and right. If they satisfy a special kind of the Riesz decomposition property, they are intervals in unital po-groups. The general comparability property in unital po-groups with strong unit (G, u), allows to compare elements of G in some intervals with Boolean ends. Such a po-group is always an ℓ-group admitting a state. We prove that every such (G, u) is a subdirect product of linearly ordered unital po-groups.

KEY WORDS: unital group; pseudo-effect algebra; general comparability; state; subdirect product.
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1. INTRODUCTION

The Abelian po-group $B(H)$, the system of all Hermitian operators of a Hilbert space H, plays an important role in orthodox quantum mechanics and in its axiomatization. The identity operator I of H is a strong unit of the po-group $B(H)$, and the interval $E(H) := \{A \in B(H) : 0 \le A \le I\}$ forms a most important example of effect algebras (Dvurečenskij and Pulmannová, 2000). Effect algebras were introduced in the 1990s by Foulis and Bennett (1994) as a $+$-counterpart of D-posets introduced by Kôpka and Chovanec (1994). Some effect algebras have an intimate connection with unital po-groups as an interval whenever they satisfy the Riesz decomposition property. Such a property is an analogue of the distributivity, however, $B(H)$ does not have the Riesz decomposition property.

Foulis (preprint, 2003, in press) studied compressions and compressible Abelian groups as well as compressible groups with two special kinds of general comparability. Such groups contain $B(H)$.

Recently, pseudo-effect algebras were introduced by me and Vetterlein (Dvurečenskij and Vetterlein, 2001a,b). They are also intervals in (non-commutative) unital po-groups when they satisfy a generalized form of the Riesz decomposition property (Dvurečenskij and Vetterlein, 2001b).

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In the present paper, we study the general comparability property in unital po-groups which are not necessary commutative. Such groups admit to compare arbitrary, two elements in special intervals with Boolean ends. We show that general comparability entails that the group with the property is an ℓ-group, which is a subdirect product of linearly ordered po-groups. In addition, it admits a state.

2. PSEUDO-EFFECT ALGEBRAS AND UNIGROUPS

According to Dvurečenskij and Vetterlein (2001a,b), a partial algebra $(E; +, 0, 1)$, where $+$ is a partial binary operation and 0 and 1 are constants, is called a *pseudo-effect algebra* if, for all $a, b, c \in E$, the following holds:

(i) $a + b$ and $(a + b) + c$ exist if, and only if, $b + c$ and $a + (b + c)$ exist, and in this case $(a + b) + c = a + (b + c)$;
(ii) there is exactly one $d \in E$ and exactly one $e \in E$ such that $a + d = e + a = 1$;
(iii) if $a + b$ exists, there are elements $d, e \in E$ such that $a + b = d + a = b + e$;
(iv) if $1 + a$ or $a + 1$ exists, then $a = 0$.

If we define $a \leq b$ if, and only if, there exists an element $c \in E$ such that $a + c = b$, then $\leq$ is a partial ordering on E such that $0 \leq a \leq 1$ for any $a \in E$. It is possible to show that $a \leq b$ if, and only if, $b = a + c = d + a$ for some $c, d \in E$. We write $c = a/b$ and $d = b \setminus a$.

Pseudo MV-algebras are lattice pseudo-effect algebras such that $(a \setminus (a \wedge b) = (a \vee b) \setminus b$ holds for all a, b.

An element $u \in G^+$ is said to be (i) a *strong unit* if given an element $g \in G$, there is an integer $n \geq 1$ such that $g \leq nu$, (ii) *generative* if given an element $g \in G^+$, there are elements $e_1, \ldots, e_n \in E := \Gamma(G, u) := \{g \in G : 0 \leq g \leq u\}$ such that $g = e_1 + \cdots + e_n$. A *unital po-group* is a couple (G, u), where G is a po-group with strong unit u. For example, if Abelian (G, u) satisfies the Riesz interpolation property, then u is generative.

We recall that $\Gamma(G, u)$ is a pseudo-effect algebra. Dvurečenskij and Vetterlein (2001a) and Dvurečenskij (2003) proved that if a pseudo-effect alegbra E satisfies a special kind of the Riesz decomposition property, then E is isomorphic with $\Gamma(G, u)$ for some unital po-group (G, u).

Let E be a pseudo-effect algebra. A mapping $\psi : E \to K$, where K is a group, is said to be a *K-valued measure* if $a, b \in E, a + b \in E$ imply $\psi(a + b) = \psi(a) + \psi(b)$.

A unital po-group (G, u) is said to be a *unigroup* if, for any group K and any K-valued measure $\psi : \Gamma(G, u) \to K$, ψ can be extended to a group homomorphism $\hat{\psi} : G \to K$; we recall that this extension is unique.

For example, if (G, u) satisfies (RDP), then (G, u) is a unigroup (Dvurečenskij and Vetterlein, 2001a,b) and u is generative. Similarly, if (G, u) is an interpolation Abelian po-group, then (G, u) is a unigroup (Ravindran, 1996). In particular, if (G, u) is a unital ℓ-group, then (G, u) is a unigroup (Dvurečenskij, 2003).

If $B(H)$ is the system of all Hermitian operators on a Hilbert space H, then $(B(H), I)$ is a unigroup, I is generative and $B(H)$ is not an interpolation group, where I is the identity operator (Foulis, preprint). We recall that owing to Kadison's theorem, $B(H)$ is an antilattice, that is, only comparable elements in $B(H)$ have joins and meets (Luxemburg and Zaanen, 1971).

More general, if $\mathcal{A}$ is a von Neumann algebra of operators acting in a complex Hilbert space H and if $B(\mathcal{A})$ is the system of all Hermitian operators in $\mathcal{A}$, then $(B(\mathcal{A}), I)$ is a unigroup (Foulis, preprint).

3. CENTRAL ELEMENTS, GENERAL COMPARABILITY, AND UNIGROUPS

An element e of a pseudo-effect algebra E is said to be *central* (or *Boolean*) if there exists an isomorphism

$$f_e : E \to [0, e] \times [0, e^{\sim}] \tag{1}$$

such that $f_e(e) = (e, 0)$ and if $f_e(x) = (x_1, x_2)$, then $x = x_1 + x_2$ for any $x \in E$.

We denote by $C(E)$ the set of all central elements of E, and $C(E)$ is said to be the *center* of E. We recall that $0, 1 \in C(E)$; in addition (Dvurečenskij, 2003), (i) if $e \in C(E)$, then $e^{\sim} = e^{-}$, we denote $e' = e^{\sim}$; (ii) $C(E) = (C(E); \vee, \wedge, ', 0, 1)$ is a Boolean algebra; (iii) if $x \in E$ and $e \in C(E)$, then $x \wedge e \in E$; (iv) if $\{e_i\}_{i=1}^n$ is a finite system of central elements of E such that $e_i \wedge e_j = 0$ for $i \neq j$ and $e_1 \vee \cdots \vee e_n = 1$, then for any $x \in E, x = x \wedge e_1 + \cdots + x \wedge e_n$; (v) if E is with (RDP), then $e \in C(E)$ iff $e \wedge e^{\sim} = 0$, or equivalently, iff $e \wedge e^{-} = 0$, and (vi) the mappings $p_e : E \to [0, e]$ and $p_{e'} : E \to [0, e']$ defined by $p_e(x) = e \wedge x$, and $p_{e'}(x) = x \wedge e'$, $x \in E$, are surjective homomorphisms such that $f_e(x) = [p_e(x), p_{e'}(x)]$ for any $x \in E$.

Suppose that $E = \Gamma(G, u)$ and (G, u) is a unigroup. Since each mapping $p_e : E \to [0, e] \subseteq G$ $(e \in C(E))$ is a homomorphism, it is also a G-valued measure. Therefore, p_e can be extended to a (unique) group homomorphism, $\hat{p}_e$, from G into G. We recall that (i) $\hat{p}_e(x) \geq 0$ for any $x \in G^+$, (ii) $\hat{p}_e(x) \leq \hat{p}_e(y)$ if $x \leq y$, (iii) $\hat{p}_e \circ \hat{p}_e = \hat{p}_e$.

Let (G, u) be a unital po-group. For any element $e \in G^+$, we denote by $G(e)$ the directed convex subgroup of G generated by e. Then, $G(e) = \bigcup_n \{g \in G : -ne \leq g \leq ne\}$.

Proposition 3.1. *Let (G, u) be a unigroup with generative u and let e be a central element of $E = \Gamma(G, u)$.*

(i) $\hat{p}_e(x) + \hat{p}_{e'}(x) = x = \hat{p}_{e'}(x) + \hat{p}_e(x)$ *for any* $x \in G$.
(ii) *If* $x \in G^+$ *and* $x \leq nu$ *for some integer* $n \geq 1$, *then* $\hat{p}_e(x) = ne \wedge x$.
(iii) $ne \wedge ne' = 0$ *for any* $n \geq 1$.
(iv) $\hat{p}_e \circ \hat{p}_{e'} = 0 = \hat{p}_{e'} \circ \hat{p}_e$.
(v) $\hat{p}_e(G) = G(e)$ *and* $\hat{p}_{e'}(G) = G(e')$ *are po-groups with strong unit* e *and* e', *respectively, and* $G = \hat{p}_e(G) \oplus \hat{p}_{e'}(G)$.
(vi) *The mapping* $f_e : G \to G(e) \times G(e')$ *given by* $f_e(x) = (\hat{p}_e(x), \hat{p}_{e'}(x))$, $x \in G$, *is a po-group isomorphism such that (a)* $f_e(e) = (e, 0)$, *(b)* $f_e(u) = (e, e^{\sim})$, *and (c) if* $f_e(x) = (x_1, x_2)$, *then* $x = x_1 + x_2$, $x \in G$.

Proof:

(i) If $x \in G^+$, then $x = x_1 + \cdots + x_n$ where $x_1, \ldots, x_n \in E$. Then $\hat{p}_e(x) + \hat{p}_{e'}(x) = x_1 \wedge e + \cdots + x_n \wedge e + x_1 \wedge e' + \cdots + x_n \wedge e' = x_1 \wedge e + x_1 \wedge e' + x_2 \wedge e + \cdots + x_n \wedge e + x_2 \wedge e' + \cdots + x_n \wedge e' = \cdots = x_1 \wedge e + x_1 \wedge e' + \cdots + x_n \wedge e + x_n \wedge e' = x_1 + \cdots + x_n = xn = \hat{p}_{e'}(x) + \hat{p}_e(x)$. The general case is now clear.
(ii) Assume $0 \leq x \leq nu$. Then $\hat{p}_e(x) \leq \hat{p}_e(nu) = np_e(u) = ne$. In addition, the monotonicity of $\hat{p}_{e'}$ implies $0 \leq \hat{p}_{e'}(x) = x - \hat{p}_e(x)$ which gives $\hat{p}_e(x) \leq x$. Let $y \in G$ with $y \leq x$ and $y \leq ne$ be given. Then $\hat{p}_e(y) \leq \hat{p}_e(x)$ and $y - \hat{p}_e(y) = \hat{p}_{e'}(y) \leq \hat{p}_{e'}(ne) = 0$, i.e., $y \leq \hat{p}_e(y) \leq \hat{p}_e(x)$ which yields $\hat{p}_e(x) = x \wedge ne$.
(iii) According to (ii), we have $ne \wedge ne' = \hat{p}_{e'}(ne) = n\hat{p}_{e'}(e) = 0$.
(iv) Let $x = x_1 + \cdots + x_n$, $x_1, \ldots, x_n \in E$. Then we have $\hat{p}_e(\hat{p}_{e'}(x)) = \sum_{i=1}^n \hat{p}_e(\hat{p}_{e'}(x_i)) = 0$.
If $x = x^+ - x^-$, where $x^+, x^- \in G^+$, then $\hat{p}_e(\hat{p}_{e'}(x)) = \hat{p}_e(\hat{p}_{e'}(x^+)) - \hat{p}_e(\hat{p}_{e'}(x^-)) = 0$.
(v If $x \in \hat{p}_e(G) \cap \hat{p}_{e'}(G)$, then $x = \hat{p}_e(x_1) = \hat{p}_{e'}(x_2)$ for some $x_1, x_2 \in G$. Therefore, $x = \hat{p}_e(x) + \hat{p}_{e'}(x) = \hat{p}_e(\hat{p}_{e'}(x_2)) + \hat{p}_{e'}(\hat{p}_e(x_1)) = 0$. In addition, from the construction of $G(e)$ and $G(e')$, we have that $\hat{p}_e$ and $\hat{p}_{e'}$ map G onto $G(e)$ and $G(e')$, respectively.
(vi) Suppose $f_e(x) \leq f_e(y)$. Then by (i), $x = \hat{p}_e(x) + \hat{p}_{e'}(x) \leq \hat{p}_e(y) + \hat{p}_{e'}(y) = y$, which proves that f_e is a po-group isomorphism of G and $G(e) \times G(e')$. □

We say that a pseudo-effect algebra E satisfies *general comparability* if, given $x, y \in E$, there is a central element $e \in E$ such that $p_e(x) \leq p_e(y)$ and $p_{e'}(x) \geq p_{e'}(y)$. This means that the coordinates of the elements $x = (p_e(x), p_{e'}(x))$ and $y = (p_e(y), p_{e'}(y))$ can be compared in $[0, e]$ and $[0, e']$, respectively.

For example, (i) every linearly ordered pseudo-effect algebra trivially satisfies general comparability; (ii) also any Cartesian product of linearly ordered pseudo-effect algebras; (iii) every σ-complete pseudo MV-algebra satisfies general comparability (Dvurečenskij, in press, Proposition 4.1).

We say that a unigroup (G, u) satisfies *general comparability* if, given $x, y \in G$, there is a central element $e \in E$ such that $\hat{p}_e(x) \leq \hat{p}_e(y)$ and $\hat{p}_{e'}(x) \geq \hat{p}_{e'}(y)$.

It is clear that if (G, u) satisfies general comparability, it satisfies $E = \Gamma(G, u)$. If (G, u) is an ℓ-group, the both notions are equivalent as shown by Jakubík (2002). In what follows, we show that general comparability in E and in the corresponding unigroup (G, u) are equivalent.

Theorem 3.2. *Let (G, u) be a unigroup and let $E = \Gamma(G, u)$. Then E satisfies general comparability, if and only if (G, u) satisfies general comparability. In such case, E is a pseudo MV-algebra and G is an ℓ-group.*

Proof: Let E satisfy general comparability. In the following steps, we prove that E is a lattice which is in fact a pseudo MV-algebra. Let $x, y \in E$ and let $e \in C(E)$ such that $p_e(x) \leq p_e(y)$ and $p_{e'}(x) \geq p_{e'}(y)$. Then $x = p_e(x) + p_{e'}(x) \geq p_e(x) + p_{e'}(y) =: v \in E$.

Claim 1. $v = x \wedge y$. We have $y = p_e(y) + p_{e'}(y) \geq p_e(x) + p_{e'}(y) = v$, that is, $v \leq x, y$. Let $z \leq x, y$. Then $p_e(z) \leq p_e(x)$ and $p_{e'}(z) \leq p_{e'}(y)$, that is, $z = p_e(z) + p_{e'}(z) \leq p_e(x) + p_{e'}(y) = v$, that is, $v = x \wedge y$.

Claim 2. $w := p_e(y) + p_{e'}(x) \in E$ and $w = x \vee y$. Since $p_e(y) \wedge p_{e'}(x) = 0$, then $w := p_e(y) + p_{e'}(x) \in E$. We conclude now $x \vee y = w$. We have $x = p_e(x) + p_{e'}(x) \leq p_e(y) + p_{e'}(x) = w$ and $y = p_e(y) + p_{e'}(y) \leq p_e(y) + p_{e'}(x) = w$. If now $z \geq x, y$, then $p_e(z) \geq p_e(y)$ and $p_{e'}(z) \geq p_{e'}(x)$ that is, $z = p_e(z) + p_{e'}(z) \geq w$.

Claim 3. $x \setminus (x \wedge y) = (x \vee y) \setminus y$ and $y \setminus (x \wedge y) = (x \vee y) \setminus x$.

Calculate

$$p_e(x \setminus (x \wedge y)) = p_e(x \setminus (p_e(x) + p_{e'}(y))) = p_e(x) \setminus p_e(x) = 0,$$
$$p_{e'}(x \setminus (x \wedge y)) = p_{e'}(x) \setminus p_{e'}(y),$$
$$p_e(y \setminus (x \wedge y)) = p_e(y) \setminus p_e(x),$$
$$p_{e'}(y \setminus (x \wedge y)) = p_{e'}(y) \setminus p_{e'}(y) = 0,$$
$$p_e((x \vee y) \setminus x) = p_e((p_e(y) + p_{e'}(x)) \setminus x) = p_e(y) \setminus p_e(x),$$
$$p_{e'}((x \vee y) \setminus x) = p_{e'}(x) \setminus p_{e'}(x) = 0,$$

$$p_e((x \vee y) \setminus y) = p_e(y) \setminus p_e(y) = 0,$$
$$p_{e'}((x \vee y) \setminus y) = p_{e'}(x) \setminus p_{e'}(y),$$

which proves Claim 3.

Finally, according to Dvurečenskij and Vetterlein (2001b, Proposition 8.7), Claim 3 is a necessary and sufficient condition to convert E into a pseudo MV-algebra $(E; \oplus, ^-, ^\sim, 0, 1)$; we define

$$a \oplus b := (a^\sim \setminus (a^\sim \wedge b))^-, \quad a, b \in E.$$

In such the case, the original $+$ and the derived one from $\oplus$ coincide.

According to the basic representation of pseudo MV-algebras as the intervals in a unital ℓ-group (Dvurečenskij, 2003), the unital po-group (G, u) is the corresponding representation ℓ-group. Applying the result of Jakubík (2002), we can show that (G, u) satisfies general comparability. As a matter of particular interest, we present the main steps of that proof. Let $x, y \in G$ and put $z = x \wedge y$, $x' = x - z$, $y' = y - z$. Then $x' \wedge y' = 0$. If we denote $x_0 = x' \wedge u$ and $y_0 = y' \wedge u$, then $x_0 \wedge y_0 = 0$. There exists an integer n such that $x' \vee y' \leq nu$. The Riesz interpolation property holding in G implies that there exist $x_1, \ldots, x_n, y_1, \ldots, y_n \in E$ such that $x' = x_1 + \cdots + x_n$ and $y' = y_1 + \cdots + y_n$. In view of general comparability holding in E, there exists $e \in C(E)$ such that $p_e(x_0) \leq p_e(y_0)$ and $p_{e'}(x_0) \geq p_{e'}(y_0)$. Therefore, $\hat{p}_e(x_0) \leq \hat{p}_e(y_0)$ and $\hat{p}_{e'}(x_0) \geq \hat{p}_{e'}(y_0)$. Consequently, $\hat{p}_e(x_0) \wedge \hat{p}_e(y_0) = 0$ and $\hat{p}_e(x_0) \wedge \hat{p}_e(y_0) = 0$. This implies $\hat{p}_e(x_0) = 0 = \hat{p}_{e'}(y_0)$. Since $x_i \leq x'$ and $x_i \leq u$ which gives $x_i \leq x_0$. Therefore, $\hat{p}_e(x_i) = 0$ and $\hat{p}_e(x') = 0$. This implies $\hat{p}_e(x') \leq \hat{p}_e(y')$.

In an analogous way, we can prove $\hat{p}_{e'}(x') \geq \hat{p}_{e'}(y')$. Taking into account that $x = x' + z$ and $y = y' + z$, we have $\hat{p}_e(x) = \hat{p}_e(x') + \hat{p}_e(z)$ and $\hat{p}_e(y) = \hat{p}_e(y') + \hat{p}_e(z)$. Finally, $\hat{p}_e(x) \leq \hat{p}_e(y)$ and $\hat{p}_{e'}(x) \geq \hat{p}_{e'}(y)$ which proves that (G, u) satisfies general comparability. □

4. CENTRAL ELEMENTS OF UNITAL PO-GROUPS

An element $e \in \Gamma(G, u)$ is said to *central* of a unital po-group (G, u) if there exists a po-group isomorphism

$$f_e : G \to G(e) \times G(e^\sim) \tag{2}$$

such that (i) $f_e(e) = (e, 0)$, (ii) $f_e(u) = (e, e^\sim)$, and (iii) if $f_e(x) = (x_1, x_2)$, then $x = x_1 + x_2$, $x \in G$. It is evident that if f_e is a po-group homomorphism from G onto $G(e) \times G(e^\sim)$ satisfying (i)–(iii), then f_e is a po-group isomorphism.

Let π_e and $\pi_{e^\sim}$ be the projection from $G(e) \times G(e^\sim)$ onto $G(e)$ and $G(e^\sim)$, respectively. Then $\phi_e : \pi_e \circ f_e$ and $\phi_{e^\sim} := \pi_{e^\sim} \circ f_e$ are po-group homomorphisms of G onto $G(e)$ and $G(e^{\sim})$, respectively, and

$$x = \phi_e(x) + \phi_{e^{\sim}}(x), \quad x \in G. \tag{3}$$

We denote by $C(G, u)$ the set of all central elements of (G, u). Then $0, u \in C(G, u)$, and $C(B(H), I) = \{0, I\}$.

Proposition 4.1. *Let e be a central element of a unital po-group (G, u) and let f_e be the po-group isomorphism from (4.1). Then*

(i) $f_e(e^{\sim}) = (0, e^{\sim})$.
(ii) *If $x \in G(e)$, then $f_e(x) = (x, 0)$. In addition, $\phi_e \circ \phi_e = \phi_e$.*
(iii) *If $y \in G(e^{\sim})$, then $f_e(y) = (0, y)$. In addition, $\phi_{e^{\sim}} \circ \phi_{e^{\sim}} = \phi_{e^{\sim}}$.*
(iv) *$G(e) \cap G(e^{\sim}) = \{0\}$. In addition, $\phi_e \circ \phi_{e^{\sim}} = 0 = \phi_{e^{\sim}} \circ \phi_e$.*
(v) $e^- = e^{\sim}$.
(vi) *For any $x \in G^+$ such that $x \leq nu$ for some $n \geq 1$, then*

$$f_e(x) = (x \wedge ne, x \wedge ne^{\sim}).$$

(vii) *$e \in C(E)$, where $E = \Gamma(G, u)$.*
(viii) *$ne \wedge ne^{\sim} = 0$ for any $n \geq 1$.*
(ix) *If $x \in G^+$ and $f_e(x) = (x_1, x_2)$, then $x_1 \vee x_2 = x$ and $x_1 \wedge x_2 = 0$.*
(x) *If $f_e(x) = (x_1, x_2)$, then $x_1 + x_2 = x = x_2 + x_1$.*
(xi) *If $f \in C(G, u)$, then $e \wedge f \in G$ and $n(e \wedge f) = ne \wedge nf$ for any $n \geq 1$.*
(xii) *$ne' = nu - ne$ for any $n1$.*
(xiii) *If $0 \leq x \leq nu$, then $x - (x \wedge ne) = (x \vee ne) - ne = -ne + (x \vee ne) = (x \wedge ne') = -(x \wedge ne) + x$.*

Proof:

(i) $f_e(e^{\sim}) = f_e(-e + u) = -f_e(e) + f_e(u) = -(e, 0) + (e, e^{\sim}) = (0, e^{\sim})$.
(ii) We recall that the element e is a strong unit in $G(e)$. Let $x \in G(e)^+$. Then $x \leq ne$ for some integer $n \geq 1$. Then $(0, 0) \leq f_e(x) \leq (x_1, x_2) \leq f_e(ne) = (ne, 0)$. Therefore, $x_2 = 0$ and $f_e(x) = (x, 0)$. If now $x \in G(e)$, then $x = x_1 - x_2$ where $x_1, x_2 \in G(e)^+$. Hence, $f_e(x) = f_e(x_1) - f_e(x_2) = (x_1, 0) - (x_2, 0) = (x, 0)$.
(iii) The proof is same as that of (ii).
(iv) If $x \in G(e) \cap G(e^{\sim})$, according to (ii) and (iii), we have $f_e(x) = (x, 0) = (0, x)$ which gives $0 = x$.
(v) $(e, e^{\sim}) = f_e(u) = f_e(e^- + e) = f_e(e^-) + (e, 0) = (e_1, e_2) + (e, 0) = (e_1 + e, e_2)$ which yields $e = e_1 + e$ and $e_2 = e^{\sim}$. Therefore, $e_1 = 0$ and $f_e(e^-) = (0, e^{\sim})$ which gives $e^- = 0 + e^{\sim} = e^{\sim}$.
(vi) By (ii) we have $0 \leq \phi_e(x) \leq x$ and $\phi_e(x) \leq \phi_e(nu) = ne$. If now $y \leq x, ne$, then $\phi_e(y) \leq \phi_e(x)$. Moreover, $-\phi_e(y) + y = \phi_{e^{\sim}}(y) \leq \phi_{e^{\sim}}(ne) = 0$. Hence, $y \leq \phi_e(y) \leq \phi_e(x)$. Therefore, $\phi_e(x) = x \wedge ne$.

(vii) By (vi), we have that the restriction of f_e onto E is an isomorphism of E onto $[0, e] \times [0, e^{\sim}]$.

(viii) By (i) and (vi), we have $f_e(ne^{\sim}) = (0, ne^{\sim}) = (ne^{\sim} \wedge ne, ne^{\sim})$ which gives $ne^{\sim} \wedge ne = 0$.

(ix) It is clear that $x \leq x_1, x_2$. Let $z \geq x_1, x_2$. Then $x_1 = \phi_e(x_1) \leq \phi_e(z)$ and $x_2 = \phi_{e^{\sim}}(x_2) \leq \phi_{e^{\sim}}(z)$, so that $x = x_1 + x_2 \leq \phi_e(z) + \phi_{e^{\sim}}(z) = z$ which proves that $x = x_1 \vee x_2$.

Let now $y \leq x_1, x_2$. Then $x_1 \leq ne$ and $x_2 \leq ne^{\sim}$ for some $n \geq 1$. By (viii), $y \leq ne, ne^{\sim}$, i.e, $y \leq 0$.

(x) Calculate, $\phi_e(x_2 + x_1) = \phi_e(x_2) + \phi_e(x_1) = \phi_e(x_1) = x_1$ and $\phi_{e^{\sim}}(x_2 + x_1)$
$= \phi_{e^{\sim}}(x_2) + \phi_{e^{\sim}}(x_1) = x_2$ which proves $x_2 + x_1 = x = x_1 + x_2$.

(xi) Since $nf \leq nu$, by (vi) we have $\phi_e(nf) = nf \wedge ne = n\phi_e(f) = n(e \wedge f)$.

(xii) We have $\phi_e(ne') = ne' \wedge ne = 0, \phi_{e'}(ne') = ne', \phi_e(nu - ne) = ne - ne$
$= 0, \phi_{e'}(nu - ne) = ne'$.

(xiii) Since $x = x \wedge ne' + x \wedge ne$, then $x - (x \wedge ne) = x \wedge ne' = x + (-x \vee -ne) = (x - x) \vee (x - ne) = 0 \vee (x - ne) = (x - ne) \vee (ne - ne) = (x \vee ne) - ne$. □

In view of (v) of Proposition 2, if $e \in C(G, u)$, then we will write

$$e' := e^{-} = e^{\sim}.$$

Theorem 4.3. *Let (G, u) be a unital po-group. If $e, f \in C(G, u)$, then $e \wedge f \in E$ and $e \wedge f \in C(G, u)$, and $C(G, u) = (C(G, u); \wedge, \vee, ', 0, u)$ is a Boolean algebra.*

Proof: It is evident that $0, u \in C(G, u)$.

Let now $e \in C(G, u)$. Then $e^{\sim} = e^{-}$ and since the mapping f_e is a po-group isomorphism of G onto $G(e) \times G(e^{\sim})$, by (ix) of Proposition 2, we have that the mapping $x \mapsto (x_2, x_1)$ whenever $f_e(x) = (x_1, x_2)$ is a po-group isomorphism of G onto $G(e^{\sim}) \times G(e)$, and it corresponds to $f_{e^{\sim}}$. Hence, $e' \in C(G, u)$.

Let $e, f \in C(G, u)$.

Claim 1. $ne \wedge nf' + ne' \wedge nf + ne' \wedge nf' = ne' \vee nf' = n(e' \vee f')$ for any $n \geq 1$.

It is easy to verify that $(ne \wedge nf)^{\sim} = ne' \vee nf' = (ne \wedge nf)^{-}$. We have $nu = ne + ne' = ne \wedge nf + ne \wedge nf' + ne' \wedge nf + ne' \wedge nf'$. Then $ne \wedge nf' + ne' = ne \wedge nf' + ne' \wedge nf + ne' \wedge nf' = ne \wedge nf' + ne' \wedge nf' + ne' \wedge nf = nf' + ne' \wedge nf \geq ne' \vee nf'$.

Let $y \geq ne', nf'$. Then $0 \leq y \leq mu$ for some integer $m \geq n$ and $y \wedge me' \geq me' \geq ne'$ and $y \wedge me \geq me \wedge mf' \geq ne \wedge nf'$ which gives $y = y \wedge me + y \wedge me' = ne' + ne \wedge nf'$.

Calculate, $\phi_e(n(e' \vee f')) = n\phi_e(e \wedge f' + e' \wedge f + e' \wedge f') = n(e \wedge f')$ and $\phi_{e'}(n(e' \vee f')) = n(e' \wedge f) + n(e' \wedge f')$ which gives $n(e' \vee f') = ne' \vee nf'$.

This proves Claim.

Claim 2. $C(G, u)$ is a lattice.

Assume $x \in G^+$, $x \leq nu$. By (xi) of Proposition 2,

$$\begin{aligned} x &= x \wedge ne + x \wedge ne' \\ &= x \wedge ne \wedge nf + x \wedge ne \wedge nf' + x \wedge ne' \wedge nf + x \wedge ne' \wedge nf' \\ &= x \wedge n(e \wedge f) + x \wedge n(e \wedge f') + x \wedge n(e' \wedge f) + x \wedge n(e' \wedge f'). \end{aligned}$$

Therefore, $x \wedge n(e \wedge f) \in G(e \wedge f)$ and $x \wedge n(e \wedge f') + x \wedge n(e' \wedge f) + x \wedge n(e' \wedge f') \in G((e \wedge f)')$ and the mapping $f_{e\wedge f} : G^+ \to G(e \wedge f)^+ \times G((e \wedge f)')^+$ defined by $f_{e\wedge f}(x) = (x \wedge n(e \wedge f), x \wedge n(e \wedge f') + x \wedge n(e' \wedge f) + x \wedge n(e' \wedge f'))$, $0 \leq x \leq nu$, is a well-defined mapping which is injective, and if $f_{e\wedge f}(x) = (x_1, x_2)$ then $x = x_1 + x_2$; $f_{e\wedge f}(e \wedge f) = (e \wedge f, 0)$ and $f_{e\wedge f}(u) = (e \wedge f, (e \wedge f)')$.

In addition, $f_{e\wedge f}$ is surjective and it preserves $+$ in G^+.

If now $x \in G$, then $x = x_1 - x_2 = -y_1 + y_2$, where $x_1, x_2, y_1, y_2 \in G$. Then $y_1 + x_1 = y_2 + x_2$ which shows that $f_{e\wedge f}$ can be extended to a po-group homomorphism denoted also by $f_{e\wedge f} : G \to G(e \wedge f) \times G((e \wedge f)')$ which is a po-isomorphism in question. This proves $e \wedge f \in C(G, u)$. Therefore, we have proved Claim 2.

Claim 3. For $0 \leq x \leq nu$, $x \wedge ne = 0$ if and only if $x \leq ne'$.

Let $x \wedge ne = 0$. Then $x = x \wedge ne + x \wedge ne' = x \wedge ne'$ which gives $x \leq ne'$. Conversely, if $x \leq ne'$, then $x = x \wedge ne + x \wedge ne' = x \wedge ne + x$, i.e., $x \wedge ne = 0$.

Claim 4. If $e \wedge f = 0$, then $e + f = e \vee f = f + e$.

If $e \wedge f = 0$, then by Claim 3, $e + f \in E$ and $f + e \in E$, and $e + f \geq e \vee f \leq f + e$. Hence, $\phi_{e\vee f}(e + f) = \phi_{e\vee f}(e) + \phi_{e\vee f}(f) = e + f \leq e \vee f$. In an analogous way $f + e \leq e \vee f$.

Claim 5. If $e \wedge f = 0$ and $x \in G^+$, then $\phi_{e\vee f}(x) = \phi_e(x) + \phi_f(x) = \phi_f(x) + \phi_e(x)$ and

$$x \wedge (ne \vee nf) = x \wedge ne + x \wedge nf = (x \wedge ne) \vee (x \wedge nf).$$

Let $0 \leq x \in nu$. Then $p_{e\vee f}(x) = x \wedge n(e \vee f)$ and $p_{e'\wedge f'}(x) = x \wedge n(e' \wedge f')$. Since $x = x \wedge ne \wedge nf + x \wedge ne \wedge nf' + x \wedge ne' \wedge nf + x \wedge ne' \wedge nf' =$

$x \wedge ne + x \wedge nf + p_{n(e \vee f)'}(x)$, so that $p_{e \vee f}(x) = x \wedge ne + x \wedge nf$. If now $z \geq x \wedge e, x \wedge f$, then $0 \leq z \leq mu$ for some integer $m \geq n$, and

$z = z \wedge me \wedge mf + z \wedge me \wedge mf' + z \wedge me' \wedge mf + z \wedge me' \wedge mf' \geq z \wedge me + z \wedge mf \geq x \wedge me + x \wedge mf = x \wedge ne + x \wedge nf$, which proves

$$x \wedge (ne \vee nf) = (x \wedge ne) + (x \wedge nf) = (x \wedge ne) \vee (x \wedge nf).$$

Claim 6. If $e \wedge f = 0, 0 \leq x \leq ne$ and $0 \leq y \leq nf$, then $x + y = y + x = x \vee y$.

It is clear that $x + y \geq x, y$. Suppose $z \geq x, y$. There exists an integer $m \geq n$ such that $0 \leq z \leq mu$. By Claim 5, we have $z \geq \phi_{e \vee f}(z) = \phi_e(z) + \phi_f(z) = z \wedge me + z \wedge mf \geq x \wedge me + y \wedge mf = x \wedge ne + y \wedge nf = x + y$. In a similar way, we prove $y + x = x \vee y$.

Claim 7. If $e \leq f$, then $e \backslash f = f \wedge e' = fe$, and $\phi_{e \wedge f'}(x) = \phi_e(x) - \phi_f(x) = -\phi_f(x) + \phi_e(x), x \in G^+$.

Since $e = f \vee e \wedge f' = f + e \wedge f'$, Claim 4 yields Claim 6.

Claim 8. If $0 \leq x \leq nu$, then

$$x \wedge (ne \vee nf) = (x \wedge ne) \vee (x \wedge nf).$$

Set $e_1 = e \wedge f', e_2 = e \wedge f$ and $e_3 = e' \wedge f$. By induction and Claims 5 and 6, we have $x \wedge (ne \vee nf) = x \wedge (ne_1 \vee ne_2 \vee ne_3) = (x \wedge ne_1) \vee (x \wedge ne_2) \vee (x \wedge ne_3) = ((x \wedge ne_1) \vee (x \wedge ne_2)) \vee ((x \wedge ne_2) \vee (x \wedge ne_3)) = x \wedge (ne_1 \vee ne_2) \vee x \wedge (ne_2 \vee ne_3) = (x \wedge ne) \vee (x \wedge nf)$.

Claim 9. $C(G, u)$ is a Boolean algebra.

By Claim 2, $C(G, u)$ is a lattice. Let $e, f, g \in C(G, u)$. If we set $x = g$, from Claim 8 we conclude $g \wedge (e \vee f) = (g \wedge e) \vee (g \wedge f)$. Passing to $'$, we have the second distributivity law. □

From the Proof of Theorem 4.2 we have that if $e, f \in C(G, u)$, then

$$\phi_{e \wedge f} = \phi_e \circ \phi_f = \phi_f \circ \phi_e.$$

In the following result we characterize central elements of (G, u) satisfying (RDP). We note that if E satisfies (RDP_0) and $a \wedge b = 0$ for $a, b \in E$, then $a + b, b + a, a \vee b \in E$, and $a \vee b = a + b = b + a$.

Corollary 4.1. *Let (G, u) be a unigroup with generative u and let $E = \Gamma(G, u)$. Then $C(E) = C(G, u)$.*

Proof: If e is a central element of (G, u), then evidently e is a central element for E.

Conversely, let e be a central element for E. According to (vi) of Proposition 1, e is a central element also for (G, u). □

Theorem 4.4. *Let a unital po-group* (G, u) *satisfy (RDP). Then* $e \in E = \Gamma(G, u)$ *is central if and only if* $e \wedge e^{\sim} = 0$ *if and only if* $e \wedge e^{-} = 0$.

Proof: Let $e \in C(G, u)$, then $e \wedge e^{\sim} = 0 = e \wedge e^{-}$. In view of (viii) of Proposition 2, $e \wedge e^{\sim} = 0$ if and only if $e \wedge e^{-} = 0$.

Conversely, let $e \wedge e^{\sim} = 0$. Then $x \le 1 = e + e^{\sim}$ for any $x \in E$. There are $x_1 \le e$ and $x_2 \le e^{\sim}$ such that $x = x_1 + x_2$. We show that if $y_1 \le e$ and $y_2 \le e^{\sim}$ and $x = y_1 + y_2$, then $x_1 = y_1$ and $x_2 = y_2$. Due to (RDP), there are four elements $c_{11}, c_{12}, c_{21}, c_{22} \in E$ such that $x_1 = c_{11} + c_{12}, x_2 = c_{21} + c_{22}, y_1 = c_{11} + c_{21}$ and $y_2 = c_{12} + c_{22}$. Since $c_{12} \le x_1 \le e$ and $c_{12} \le y_2 \le e^{\sim}$, we conclude $c_{12} = 0$. Similarly, $c_{21} = 0$. Hence, $x_1 = c_{11} = y_1$ and $x_2 = c_{22} = y_2$.

Define the mapping $p_e : E \to [0, e]$ by $p_e(x) = x_1$ if $x = x_1 + x_2$ $(x \in E)$. If $x_1 \in [0, e]$ and $x_2 \in [0, e^{\sim}]$, then $x_1 \wedge x_2 = 0$, so that by the earlier note, $x = x_1 + x_2 = x_2 + x_1 = x_1 \vee x_2$, and hence $p_e(x) = x_1$. Consequently, p_e restricted to $[0, e]$ is the identity.

We show that p_e is a homomorphism of pseudo-effect algebras. Let $x + y \in E$ and $x = x_1 + x_2$ and $y = y_1 + y_2$, where $x_1, y_1 \le e, x_2, y_2 \le e^{\sim}$. Then $x + y = x_1 + x_2 + y_1 + y_2$. Since $x_2 \wedge y_1 = 0$, then $x + y = x_1 + y_1 + x_2 + y_2$. On the other hand, let $x + y = z_1 + z_2$, where $z_1 \le e$ and $z_2 \le e^{\sim}$. Hence, there are four elements $d_{11}, d_{12}, d_{21}, d_{22}$ such that

$$x_1 + y_1 = d_{11} + d_{12},$$

$$x_2 + y_2 = d_{21} + d_{22},$$

$$z_1 = d_{11} + d_{21},$$

$$z_2 = d_{12} + d_{22}.$$

We claim $d_{12} = 0$. Since $d_{12} \le x_1 + y_1$, then $d_{12} = d' + d''$, where $d' \le x_1$ and $d'' \le y_1$. Then $d' \le x_1 \le e$ and $d' \le d_{12} \le z_2 \le e^{\sim}$ so that $d' = 0$, and $d'' \le y_1 \le e$ and $d'' \le z_2 \le e^{\sim}$ proving $d'' = 0$ and therefore $d_{12} = 0$. In a similar way, we can prove $d_{21} = 0$ which yields $x_1 + y_1 = z_1$ and $x_2 + y_2 = z_2$, so that, p_e is a homomorphism.

By the earlier note, we have $e^{\sim} = e^{-}$. Therefore, we can write $e' := e^{\sim} = e^{-}$, and let $p_{e'}(x) = x_2$ if $x = x_1 + x_2$ $(x \in E)$. Then $p_{e'}$ is a homomorphism from E onto $[0, e']$.

Consequently, the mapping $f_e : E \to [0, e] \times [0, e']$ defined by $f_e(x) = (p_e(x), p_{e'}(x))$, $x \in E$, is an isomorphism of pseudo-effect algebras with

$f_e(e) = (e, 0)$, so that $e \in C(E)$. Since (G, u) satisfy (RDP), (G, u) is a unigroup with generative u, by Corollary 1, $e \in C(G, u)$. □

We say that a po-group G is *Dedekind monotone σ-complete* if any sequence $x_1 \le x_2 \le \cdots$ in G which has an upper bound, $x \in G$, has a supremum $\bigvee_{n=1}^{\infty} x_n$ in G. We recall that a Dedekind monotone po-group is not necessarily a lattice (Goodearl, 1986, Examples 16.1 and 16.8), or the unital po-group $B(H)$ which is an antilattice (Luxemburg and Zaanen, 1971).

Another example, let $G = \mathbb{Z}^2$ with the strict ordering $\le$, that is, $(m_1, n_1) \le (m_2, n_2)$ iff either $m_1 < m_2$ and $n_1 < n_2$ or $(m_1, n_1) = (m_2, n_2)$. Then (G, u), where $u = (1, 1)$, is a unital po-group which not an interpolation group. (G, u) is Dedekind monotone σ-complete (all bounded ascending or descending sequences from $\mathbb{Z}^2$ are eventually constant).

Theorem 4.5. *Let a unital po-group (G, u) be Dedekind monotone σ-complete such that if $0 \in G^+$ is the infimum of two elements, x and y, from G^+, then 0 is the infimum of x and y also in G. Let $e = \bigvee_{i=1}^{\infty} e_i \in G$, where $e_i \in C(G, u)$, $i \ge 1$. Then $e \in C(G, u)$, and for any $n \ge 1$ and any $0 \le x \le nu$*

$$x \wedge \left(\bigvee_{i=1}^{\infty} ne_i \right) = \bigvee_{i=1}^{\infty} (x \wedge ne_i). \tag{4}$$

Proof: Since by Theorem 4.2, $C(G, u)$ is a Boolean algebra, without loss of generality, we can assume $e_1 \le e_2 \le \cdots$. Therefore, $e \in G$. We recall that we also have $e^{\sim} = e^{-} =: e'$.

In addition, $x \wedge e_i \in E$, which entails

$$x_0 := \bigvee_i (x \wedge ne_i)$$

is defined in G, and $x_0 \le x, e$.

Claim 1. $ne = \bigvee_i ne_i$ *for any* $n \ge 1$.

Indeed, for simplicity assume $n = 2$. Then $e + e = \left(\bigvee_{i=1}^{\infty} e_i\right) + e = \bigvee_{i=1}^{\infty} (e_i + e) = \bigvee_{i=1}^{\infty} \bigvee_{j=1}^{\infty} (e_i + e_j) = \bigvee_{i=1}^{\infty} 2e_i$.

Assume that x_0^* is any element of G such that $x \wedge ne_i \le x_0^* \le x, ne$ for any i; such an element always exists, e.g., x_0.

Claim 2. $\bigwedge_i (x - (x \wedge ne_i)) = x - x_0^* = -x_0^* + x$.

It is evident that $x - (x \wedge ne_i) \ge x - x_0^*$ for every i. Let $d \le x - (x \wedge ne_i)$ for each i. By (xiii) of Proposition 2, $d \le x - (x \wedge ne_i) = (x \vee ne_i) - ne_i$. Then $d + ne_i \le x \vee ne_i \le x - x_0^* + ne$ and $ne_i \le -d + x - x_0^* + ne$ which gives $ne \le -d + x - x_0^* + ne$ which gives $d \le x - x_0^*$.

Claim 3. $x_0 = \bigvee_i (x \wedge ne_i) = x_0^*$.

From Claim 2, we have $\bigwedge_i (x - (x \wedge ne_i)) = x - \bigvee_i (x \wedge ne_i) = x - x_0^*$, that is, $\bigvee_i (x \wedge ne_i) = x_0^*$.

Claim 4. $(x - x_0^*) \wedge (ne - x_0^*) = 0$.

Assume $0 \le z \le x - x_0^*$ and $z \le ne - x_0^*$. Then $z + x_0^* \le x$, $z + x_0^* \le ne$, and $x \wedge ne_i \le z + x_0^* \le ne, x$ for each i. Using Claim 3, we have $x_0^* = z + x_0^*$, that is, $z = 0$.

Claim 5. $ne \wedge ne' = 0$ *for any* $n \ge 1$.

Assume $0 \le z \le ne, ne'$. For $z_0 := \bigvee_i (z \wedge ne_i)$ we have $z_0 \le z \le ne, ne'$ and $z \wedge ne_i \le z_0 \le ne' \le ne_i'$ which gives $z \wedge ne_i = 0$ for any i, i.e., $z_0 = 0$. Then $z - z_0 \le ne - z_0$ and by Claim 3, we have $z - z_0 = (z - z_0) \wedge (ne - z_0) =$ proving $z = 0$.

Define two mappings $q_e : G^+ \to G(e)^+$ and $q_{e'} : G^+ \to G(e')^+$ by

$$q_e(x) := \bigvee_i (x \wedge ne_i) =: x_0,$$

$$q_{e'}(x) := x - x_0$$

for any $0 \le x \le nu$. q_e is well defined, while if $0 \le x \le nu$ and $x \le mu$, then $x \wedge ne_i = x \wedge me_i$ for any i. By Claims 2–3, we have $q_{e'}(x) = x - x_0 = -x_0 + x = \bigwedge_i (x \wedge ne_i') \in G(e')^+$.

Then $q_e(e) = e$ and $q_{e'}(e) = 0$.

Claim 6. If $x, y \in G^+$ such that $x + y \le nu$, then $q_e(x + y) = q_e(x) + q_e(y)$.

Calculate, $q_e(x + y) = \bigvee_i ((x + y) \wedge ne_i) = \bigvee_i (x \wedge ne_i + y \wedge ne_i) \le q_e(x) + q_e(y) \in G(e)^+$.

Assume $(x + y) \wedge ne_i \le z$ for any i, and fix an integer $i_0 \ge 1$. Then $x_0, y_0 \le z$ and $x \wedge ne_i + y \wedge e_{i_0} \le z$ for any $i \ge i_0$. Hence, $x \wedge ne_i \le z - (y \wedge ne_{i_0})$, that is, $x_0 \le z - (x \wedge ne_{i_0})$ and $y \wedge ne_{i_0} \le -x_0 + z$ which gives $y_0 \le -x_0 + z$ and $x_0 + y_0 \le z$.

Claim 7. If $x, y \in G^+$ such that $x + y \le nu$, then $q_{e'}(x + y) \ge q_{e'}(x) + q_{e'}(y)$.

Indeed, $q_{e'}(x + y) = \bigwedge_i ((x + y) \wedge ne_i') = \bigwedge_i (x \wedge ne_i' + y \wedge ne_i') \ge x \wedge e' + y \wedge e' \in G^+$.

Claim 8. If $0 \le x \le ne$, $0 \le y \le ne'$, then $q_e(x) = x$ and $q_{e'}(y) = y$.

Calculate, $q_e(x) = x_0$ and $q_{e'}(x) = x - x_0 \le e, e'$ which by Claim 5 means $x - x_0 = 0$. Similarly we prove $q_{e'}(y) = y$.

Claim 9. If $0 \le x \le ne$ and $0 \le y \le ne'$, then $x + y = x \vee y = y + x$.

Assume $z \geq x, y$. Then $q_e(z) \geq q_e(x) = x$ and $q_{e'}(z) \geq q_{e'}(y) = y$ which gives $z = q_e(z) + q_{e'}(z) \geq x + y$, that is, $x + y = x \vee y$.

We assert that $q_{e'}(x + y) = y$. Indeed, $x + y = q_e(x + y) + q_{e'}(x + y) \geq q_e(x) + q_e(y) + q_{e'}(x) + q_{e'}(y) = x + y$.

Assume now $x + y = y + d$ for some $d \in G^+$. Then $x = q_e(x + y) = q_e(y + d) = q_e(d)$ and $y = q_{e'}(x + y) = q_{e'}(y + d) \geq y + q_{e'}(d)$ which implies $x + y = y + d \geq y + q_e(d) = y + x$. But $y + x \geq x, y$, then $y + x \geq x \vee y = x + y$.

Claim 10. If $x, y \in G^+$ and $x + y \leq nu$, then $q_{e'}(x + y) = q_{e'}(x) + q_{e'}(y)$.

Calculate and use Claim 9, $x + y = q_e(x + y) + q_{e'}(x + y) \geq q_e(x) + q_e(y) + q_{e'}(x) + q_{e'}(y) = q_e(x) + q_{e'}(x) + q_e(y) + q_{e'}(y) = x + y$.

Claim 11. If $f_e : G^+ \to G(e)^+ \times G(e')^+$ is defined by

$$f_e(x) = (q_e(x), q_{e'}(x)), \quad x \in G,$$

then f_e is a $+$-preserving injective mapping onto $G(e) \times G(e')$.

Indeed, $f_e(e) = (e, 0)$, and if $f_e(x) = (x_1, x_2)$, then $x = x_1 + x_2$, and by Claims 8 and 10, f_e is an injective mapping preserving $+$. Assume $0 \leq x \leq ne$ and $0 \leq y \leq ne'$, then $x + y \leq nu$ and $f_e(x + y) = (x, y)$.

As in the Proof of Theorem 4.2, we can extend f_e to a mapping from G to $G(e) \times G(e')$ which is also denoted by f_e. It is possible to show that f_e is a po-group isomorphism, which proves that e is a central element of (G, u).

Therefore, $x \wedge ne \in E$, so that $x \wedge ne = x_0$ which proves (4.3). □

It is worth noting that the statement, "if for $x, y \in G^+$ the infimum in G^+ is 0, then the infimum of x, y taken is the whole G is also 0," is equivalent with the statement "the infimum of $a, b \in G^+$ is the same as the infimum of a, b taken in G."

We recall that if (G, u) is an ℓ-group, then infimum of two positive elements taken in G^+ is the same as that taken in the whole G. In this particular case, if in addition, G is Dedekind monotone σ-complete, then G is a Dedekind σ-complete ℓ-group. In a similar way, if (G, u) satisfies (RDP$_1$), then infimas taken in $E = \Gamma(G, u)$ are preserved also in G ((Dvurečenskij and Vetterlein, 2001b, Proposition 6.3).

5. GENERAL COMPARABILITY AND STATES ON UNITAL PO-GROUPS

Now we extend the notion of general comparability also for unital po-groups.

We say that a unital po-group (G, u) satisfies *general comparability* if, given $x, y \in G$, there is a central element $e \in C(G, u)$ such that $\phi_e(x) \leq \phi_e(y)$ and $\phi_{e'}(x) \geq \phi_{e'}(y)$.

The following result extends the result holding for Abelian unital groups with interpolation (Goodearl, 1986, Proposition 8.9), as well as that for Abelian unital compressible unital po-groups not necessarily with interpolation (Foulis, preprint, Theorem 4.9).

Theorem 5.1. *Let a unital po-group (G, u) satisfy general comparability. Then G is an ℓ-group.*

Proof: Given $x, y \in G$, there exists $e \in C(G, u)$ such that $\phi_e(x) \leq \phi_e(y)$ and $\phi_{e'}(x) \geq \phi_{e'}(y)$. Set $v = \phi_e(x) + \phi_{e'}(y)$. Then $v \leq \phi_e(y) + \phi_{e'}(y) = y$ and similarly $v \leq x$.

If now $z \leq x, y$, then $\phi_e(z) \leq \phi_e(x)$ and $\phi_{e'}(z) \geq \phi_{e'}(y)$ whence $z = \phi_e(z) + \phi_{e'}(z) \leq \phi_e(x) + \phi_{e'}(y) = v$ which proves $v = x \wedge y$.

In a similar way, we can prove that if $w := \phi_e(y) + \phi_{e'}(x)$, then $w = x \vee y$. □

Comparing Theorems 1 and 3, we see that if a unital po-group (G, u) satisfies general comparability, then it is a unigroup which is an ℓ-group. In addition, if (G, u) is a Dedekind σ-complete ℓ-group, then (G, u) is commutative and it satisfies general comparability (Goodearl, 1986, Theorem 9.9). This and Theorem 1 prove the following corollary.

Corollary 5.2. *Let (G, u) satisfy general comparability. Then $C(G, u) = C(E)$, where $E = \Gamma(G, u)$.*

We recall that a *state* on a unital po-group (G, u) is any mapping $\hat{s} : G \to \mathbb{R}$ such that (i) $\hat{s}(g) \geq$ for any $g \in G^+$, (ii) $\hat{s}(g + h) = \hat{s}(g) + \hat{s}(h)$ for all $g, h \in G$, and (iii) $\hat{s}(u) = 1$. It is well known that if (G, u) is an Abelian group, then (G, u) admits a state (Goodearl, 1986, Corollary 4.4) whenever $u > 0$. In contrast that, for non-commutative unital po-groups this is not always the case, as it was shown in (Dvurečenskij, 2001), or see the following example.

We apply similar notations as in (Glass, 1999). Let $\mathbb{R}$ be the set of all real numbers with the natural linear order. We denote by $A(\mathbb{R})$ the set of all order-preserving permutations of $\mathbb{R}$. Then $A(\mathbb{R})$ is a group under composition. For $f, g \in A(\mathbb{R})$ we put $f \leq g$ if $f(t) \leq g(t)$ for each $t \in \mathbb{R}$. The relation $\leq$ is a partial order on $A(\mathbb{R})$ and under this partial order, $A(\mathbb{R})$ turns out to be a lattice ordered group.

Example 5.3. Let $a \in A(\mathbb{R})$, $a(t) \geq t$ for any $t \in \mathbb{R}$, and $a(t_0) > t_0$ for some $t_0 \in \mathbb{R}$. Then (G_a, a) is a stateless unital ℓ-group, where G_a denotes the convex ℓ-subgroup of $A(\mathbb{R})$ generated by the element a. In addition, general comparability fails to hold in M (see Theorem 4).

In what follows we show that if (G, u) satisfies general comparability, then it admits a state.

Theorem 5.4. *Every unital po-group satisfying general comparability admits a state.*

Proof: Let (G, u) be a unital po-group satisfying general comparability. According to Theorem 4, G is an ℓ-group and $E = \Gamma(G, u)$ is a pseudo MV-algebra such that $C(E) = C(G, u)$ and E satisfies general comparability. Applying (Dvurečenskij; Cor 4.5], E admits a state, s. Since (G, u) is a unigroup, s can be extended to a state, $\hat{s}$, on (G, u). □

In what follows we show that if (G, u) satisfies general comparability, then G is representable, that is, it is a subdirect product of linearly ordered unital po-groups.

Let $\{(M_t; \oplus_t, ^{-_t}, ^{\sim_t}, 0_t, 1_t)\}_{t\in T}$ be a family of pseudo MV-algebras. The Cartesian product $M := \prod_{t\in T} M_t$, where $\oplus, ^-, ^\sim, 0, 1$ are defined in a usual way by coordinates, is said to be a *direct product* of $\{(M_t; \oplus_t, ^{-_t}, ^{\sim_t}, 0_t, 1_t)\}_{t\in T}$. Then M is a pseudo MV-algebra. A pseudo MV-algebra M is a *subdirect product* of a family of $\{(M_t; \oplus_t, ^{-_t}, ^{\sim_t}, 0_t, 1_t)\}_{t\in T}$ of pseudo MV-algebras iff there exists a one-to-one homomorphism $h : M \to \prod_{t\in T} M_t$ of pseudo MV-algebras such that, for each $t \in T$, $\pi_t \circ h$ is a homomorphism of pseudo MV-algebras from M onto M_t, where π_t is the t-th projection $\prod_{t\in T} M_t$ onto M_t.

According to (Georgescu and Iorgulescu, 2001), we say that a pseudo MV-algebra M is *representable* if it can be represented as a subdirect product of linear pseudo MV-algebras. It is well known that every MV-algebra is representable (see e.g., Cignoli et al., 2002).

An ℓ-group G is representable iff it is a subdirect product of linearly ordered ℓ-groups, or equivalently (Darnel, 1995, Proposition 47.1(c)), iff there exists a system of ℓ-ideals, $\{L_t : t \in T\}$, of G such that $\bigcap_{t\in T} L_t = \{0\}$.

Theorem 5.5. *Every unital po-group satisfying general comparability is a representable as a subdirect product of linearly ordered unital po-groups.*

Proof: Let (G, u) be a unital po-group satisfying general comparability. Then (G, u) is an ℓ-group, and $E = \Gamma(G, u)$ is a pseudo MV-algebra with general comparability. According to (Dvurečenskij) Thm 6.2], E is representable, and this is possible iff G is representable, see (Dvurečenskij, 2001). If E is a subdirect product of a system of linearly ordered pseudo MV-algebras, $\{E_i\}$, then according to basic representation of pseudo MV-algebras (Dvurečenskij, 2003), every $E_i = \Gamma(G, u_i)$, where (G_i, u_i) is a linearly ordered unital ℓ-group. Hence, G is a subdirect product of $\{G_i\}$. □

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